Get It Solved 2019 Contest. Mathematics

Answers and Solutions

1. After summarizing the contest "Get IT Solved" results it has turned out that each of the participants who solved problem #2, also solved problem #1. Furthermore, exactly 90% of those who have not solved problem #2, have not solved problem #1 either. What is the share of the participants who have solved problem #2 given that problem #1 has been solved by exactly 55% of participants?

   (The answer is an integer between 0 and 100 – the percentage of those who have solved the problem).

   Answer. 50.

   Solution. Let us denote the quantities of participants who solved and who have not solved problem #2 as a and b respectively. It follows from the task that there are exactly $0.9b$ participants who have not solved problem #1. It implies that there are $0.1b$ participants who solved problem #1 and have not solved problem #2. It is also known that $a + 0.1b$ constitutes 55% of the total number of participants that is equal to $a + b$.

   Thus we have $a+0.1b = 0.55(a+b)$, and so $0.45a = 0.45b$, $a = b$. Thus, problem #2 has been solved by exactly one half of the participants.

2. A square sheet of paper has been cut into eight square parts, the areas of seven of them being equal to 25 square centimeters each. What is the area of the eighth square?

   (Express the answer in quadratic centimeters.)

   Answer. 225.

   Solution. Let $d \times d$, $a \times a$ and $b \times b$ be dimensions of a sheet of paper, of a "small" square (each of seven of them) and of a big square respectively. Equating the areas yields $7a^2 + b^2 = d^2$.

   One of the sides of the sheet is totally laid with small squares, and therefore d is a multiple of a. Let $d = na$, $n \in \mathbb{N}$. It is easy to understand that b is also a multiple of a (you can see it by extending the sides of a big square to the edges of the sheet of paper); let $b = ma$, $m \in \mathbb{N}$. Substituting these expressions into the equality of areas and reducing it by a^2 we get $7+m^2 = n^2$, $7 = (n-m)(n+m)$.

   As 7 is a prime number and both factors on the right are positive integers, the only possibility is $n-m = 1$, $n+m = 7$. Then $n = 4$, $m = 3$, and the area of a big square equals $(ma)^2 = 3^2 \cdot 25 = 225$.

3. Angle bisector CD and altitude CH are drawn from the vertex of the right angle C of a triangle ABC. It has turned out that $AD = 2$, $BD = 1$. Find the length of altitude CH. (Express the answer as a decimal fraction rounded up to one digit after the decimal point.)

   Answer. 1.2.

   Solution. Let us denote $BC = x$, $CH = h$. The angle bisector theorem yields $BD : AD = BC : AC$, and so $1 : 2 = x : AC$, $AC = 2x$. From Pythagorean theorem we get that $BC^2 + AC^2 = AB^2$, i.e. $x^2 + 4x^2 = 3^2$, $x^2 = \frac{9}{5}$.

   The doubled area of triangle ABC is equal to $BC \cdot AC = 2x^2$. On the other hand, it can be calculated as $AB \cdot CH = 3h$. Hence, $h = \frac{2x^2}{3} = \frac{6}{5} = 1.2$.

4. Two cyclists start moving simultaneously from diametrically opposed points on a circular cycle track. Each of them goes counterclockwise with his own constant velocity. The first cyclist goes faster, and so he overtakes the second one now and then. It is known that the time that has passed from the start of the race to the second overtaking is equal to 2 minutes. Find the time that passes from the start to the moment of the fifth overtaking.

   (Express the answer in minutes.)

   Answer. 6.

Solution. Let v be the difference in the cyclists' speeds. Let us notice that it is the speed with which the fastest cyclist catches up with the other one. Then time t required for the first overtaking is equal to $\frac{d}{2} : v$ where d is the length of a track. After that, for the next overtaking to happen, the faster cyclist needs to go a distance by d greater than the slower one. It requires a time of $d : v = 2t$.

Thus the time from the start until the second overtaking is equal to $3t$, and the time until the fifth overtaking is equal to $9t$. It is known that $3t$ is equal to 2 minutes, and so $9t$ is equal to 6 minutes.

5. Island Avalon is inhabited with knights, knaves and spies. The knights always tell the truth, the knaves always tell lies, and the spies always alternate their answers (i.e. the first answer can be either a truth or a lie, and after that a true answer necessarily follows a lie, and a lie follows a true answer). Each of the inhabitants of Avalon island got three questions: "Are you a knight?", "Are you a knave?", "Are you a spy?", and each of them answered either "Yes" or "No" to each of the questions. It is known that answer "Yes" was received 100, 25 and 55 times for the first, the second and the third questions respectively. Find the largest possible number of knights on the island.

Answer. 70.

Solution. Let t and f be quantities of knights and knaves on the island. As for the spies, let us split them into two groups: those who start answering the questions with telling the truth (ST) and those who start with telling a lie (SL). Let us denote the amount of spies in each of the groups as x and y respectively.

Let us make the table that contains the answers of different groups of people to the questions.

Question	Knight's answer	Knave's answer	ST's answer	SL's answer
#1 Are you a knight?	yes	yes	no	yes
#2 Are you a knave?	no	no	yes	no
#3 Are you a spy?	no	yes	yes	no

We see that knights and spies starting with lies give the same answers to all the questions! Therefore the numbers of "yes" obtained for the three questions are equal to $t+f+y$, x and $f+x$ respectively. From the condition of the problem we get the equations: $t+f+y=100$, $x=25$, $f+x=55$.

Solving them, we find that $f=55-x=55-25=30$, $t+y=100-f=70$. So the number of spies starting with lies and knights amounts to 70. It means that the largest possible number of knights is 70.